

Global existence in critical spaces for density-dependent incompressible viscoelastic fluids

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Abstract

In this paper we consider the local and global well-posedness to the density-dependent incompressible viscoelastic fluids. We first study some linear models associated to the incompressible viscoelastic system. Then we approximate the system by a sequence of ordinary differential equations, by means of the Friedrichs method. Some uniform estimates for those solutions will be obtained. Using compactness arguments, we will get the local existence up to extracting a subsequence by means of Ascoli's lemma. With the help of small data conditions and hybrid Besov spaces, we finally derive the global existence.

1 Introduction

Elastic solids and viscous fluids are two extremes of material behavior. Viscoelastic fluids show intermediate behavior with some remarkable phenomena due to their elastic nature. Their exhibit a combination of both fluid and solid characteristics and have received a great deal interest. It can also be regarded as the consistence condition of the flow trajectories obtained from the velocity field u and also of those obtained from the deformation tensor F . Classically the motion of a fluid is described by a time-dependent family of orientations preserving diffeomorphism $X(t, x)$. Then deformation tensor F is defined as

$$F(t, x) = \frac{\partial X(t, x)}{\partial x}.$$

Applying the chain rule, we see that $F(t, x)$ satisfies the following transport equation (see [15]):

$$\partial_t F + u \cdot \nabla F = \nabla u \cdot F.$$

The viscoelastic fluid system of the incompressible in the Oldroyd-B model takes the following form:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = \operatorname{div}(FF^\top), \\ \partial_t F + u \cdot \nabla F = \nabla u \cdot F, \\ \operatorname{div} u = 0, \quad (u, F)|_{t=0} = (u_0, F_0), \end{cases}$$

where u is the fluid velocity, Π is the pressure and F is the deformation tensor introduced above. Recently, the system (1.1) has been studied extensively. Lin, Liu and Zhang in [14], Lei, Liu and Zhou in [13], Lin and Zhang in [16] proved the local well-posedness of (1.1) in Hilbert space H^s , and global well-posedness with small initial data. Local well-posedness can be proved by the standard energy method, while to obtain a global result, a very subtle energy estimate is applied to capture the damping mechanism on $F - I$. When one adds a linear damping term in the evolution equation of FF^\top , which is the Cauchy-Green strain tensor, Chemin and Masmoudi [6] proved the existence of a local solution and a global small

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solution in critical Besov spaces. We refer to [17] and [19] for the well-posedness of the system (1.1) in critical spaces.

In fact, in the real world, the density usually depends on time. So we are more interested in the density dependent system, which is more close to the real situation. In this paper, we want to investigate the global well-posedness for the incompressible viscoelastic fluids described by the following system:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & x \in \mathbb{R}^N, t > 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = \operatorname{div}(\det(F)^{-1} F F^\top), \\ \partial_t F + u \cdot \nabla F = \nabla u \cdot F, \\ \operatorname{div} u = 0, & (\rho, u, F)|_{t=0} = (\rho_0, u_0, F_0). \end{cases} \quad (1.1)$$

The initial data (ρ_0, u_0, F_0) are prescribed.

Throughout this paper, we will use the notations of

$$(\nabla v)_{i,j} = \frac{\partial v_i}{\partial x_j}, (\nabla v F)_{i,j} = (\nabla v)_{i,k} F_{k,j}, (\nabla \cdot F)_i = \partial_j F_{i,j},$$

and the summation over repeated indices will always be understood. We also assume that $a_0 = \frac{1}{\rho_0} - 1$, $E_0 = F_0 - I$ and E_0 satisfy the following constraints:

$$\det(E_0 + I) = 1, \operatorname{div}(E_0^\top) = 0, \quad (1.2)$$

and

$$\partial_m E_{0ij} - \partial_j E_{0im} = E_{0lj} \partial_l E_{0im} - E_{0lm} \partial_l E_{0ij}. \quad (1.3)$$

Using these constraints, we obtain that

$$\begin{cases} \det(E + I) = 1, \operatorname{div}(E^\top) = 0, \\ \partial_m E_{ij} - \partial_j E_{im} = E_{lj} \partial_l E_{im} - E_{lm} \partial_l E_{ij}, \end{cases} \quad (1.4)$$

by Proposition 1 in [18]. From the definition of F , we note that the assumption of $\det(E_0 + I) = 1$ is nature. The first two of these expressions are just the consequences of the incompressibility condition and the last one can be understood as the consistency condition for changing variables between the Lagrangian and Eulerian coordinates.

At this stage, we will use scaling considerations for (1.1) to guess which spaces may be critical. We observe that (1.1) is invariant by the transformation

$$\begin{aligned} (\rho(t, x), u(t, x), F(t, x), \Pi(t, x)) &\rightarrow (\rho(l^2 t, lx), lu(l^2 t, lx), F(l^2 t, lx), l^2 \Pi(l^2 t, lx)), \\ (\rho_0(x), v_0(x), F_0(x)) &\rightarrow (\rho_0(lx), lv_0(lx), F_0(lx)). \end{aligned}$$

Definition 1.1. A function space $E \subset (\mathcal{S}'(\mathbb{R}^N) \times \mathcal{S}'(\mathbb{R}^N))^N \times (\mathcal{S}'(\mathbb{R}^N))^{N \times N}$ is called a critical space if the associated norm is invariant under the transformation $(\rho(x), u(x), F(x)) \rightarrow (\rho(lx), lu(lx), F(lx))$.

Obviously $\dot{H}^{\frac{N}{2}} \times (\dot{H}^{\frac{N}{2}-1})^N \times (\dot{H}^{\frac{N}{2}})^{N \times N}$ is a critical space for the initial data. The space $\dot{H}^{\frac{N}{2}}$ however is not included in L^∞ , we cannot expect to get L^∞ control on the density and deformation tensor, when we choose $F_0 - I \in (\dot{H}^{\frac{N}{2}})^{N \times N}$. Moreover, the product between functions does not extend continuously from $\dot{H}^{\frac{N}{2}-1} \times \dot{H}^{\frac{N}{2}}$ to $\dot{H}^{\frac{N}{2}-1}$, so that we will run into difficulties when estimating the nonlinear terms. Similar to the compressible Navier-Stokes system [7], we could use homogeneous Besov spaces $\dot{B}_{2,1}^s(\mathbb{R}^N)$ (defined in [1], Chapter 2). $\dot{B}_{2,1}^{\frac{N}{2}}$ is an algebra embedded in L^∞ which allows us to control the density and deformation tensor from above without requiring more regularity on derivatives of ρ_0 and F_0 . From now on, we define the density and usual strain tensor by the form

$$a := \frac{1}{\rho} - 1, E := F - I.$$

Then system (1.1) can be rewritten as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, & x \in \mathbb{R}^N, t > 0, \\ \partial_t u_i + u \cdot \nabla u_i - (a+1)(\mu \Delta u_i - \nabla_i \Pi) = G_i, \\ \partial_t E + u \cdot \nabla E = \nabla u \cdot E + \nabla u, \\ \operatorname{div} u = 0, & (\rho, u, E)|_{t=0} = (a_0, u_0, E_0), \end{cases} \quad (1.5)$$

where $G_i = (a+1)(\partial_j E_{ik} E_{jk} + \partial_j E_{ij})$.

Now we can state our main results. First define the following functional spaces:

$$X_T^s = C([0, T]; B_{2,1}^{s-1}) \cap L^1([0, T]; B_{2,1}^{s+1}) \times C([0, T]; B_{2,1}^s),$$

$$Y^s = C(\mathbb{R}^+; \dot{B}_{2,1}^{s-1}) \cap L^1(\mathbb{R}^+; \dot{B}_{2,1}^{s+1}) \times C(\mathbb{R}^+; \tilde{B}_\mu^{s,\infty}).$$

Then the norm of X_T^s and Y^s are defined by

$$\|(u, E)\|_{X_T^s} = \|u\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} + \|u\|_{L_T^1(B_{2,1}^{s+1})} + \|E\|_{\tilde{L}_T^\infty(B_{2,1}^s)},$$

$$\|(u, E)\|_{Y^s} = \|u\|_{\tilde{L}^\infty(\mathbb{R}^+, \dot{B}_{2,1}^{s-1})} + \|u\|_{L^1(\mathbb{R}^+, \dot{B}_{2,1}^{s+1})} + \|E\|_{\tilde{L}^\infty(\mathbb{R}^+, \tilde{B}_\mu^{s,\infty})}.$$

Here $B_{p,r}^s$ denotes the nonhomogeneous Besov space and $\dot{B}_{p,r}^s$ denotes the homogeneous space. The hybrid Besov space $\tilde{B}_\mu^{s,\infty}$ will be defined in the following section.

Theorem 1.1 (Local well-posedness). *Suppose that initial data satisfy the incompressible constrain (1.2), $a_0 \in B_{2,1}^{\frac{N}{2}}$, $u_0 \in B_{2,1}^{\frac{N}{2}-1}$ and $E_0 \in B_{2,1}^{\frac{N}{2}}$. Then there exist $T > 0$ and a unique local solution for system (1.2) with*

$$a \in C([0, T]; B_{2,1}^{\frac{N}{2}}), \quad (u, E) \in X_T^{\frac{N}{2}} \quad \text{and} \quad \nabla \Pi \in L_T^1(B_{2,1}^{\frac{N}{2}-1}).$$

Besides, the following estimate is valid

$$\|a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} + \|(u, E)\|_{X_T^{\frac{N}{2}}} \leq C(\|a_0\|_{B_{2,1}^{\frac{N}{2}}} + \|u_0\|_{B_{2,1}^{\frac{N}{2}-1}} + \|E_0\|_{B_{2,1}^{\frac{N}{2}}}),$$

where C is a constant depending only on N and μ .

Remark 1.1. We do not need the smallness condition on a_0 compared with the assumption of R. Danchin in [9] which consider the local well-posedness in homogeneous Besov space. The method was first introduced by R. Danchin in [12] when dealing with the well-posedness of the barotropic viscous fluids in critical spaces. One can see in Section 3 that, for the technical reason, we could only study the local well-posedness on the nonhomogeneous Besov space without the smallness condition on a_0 .

Theorem 1.2 (Global well-posedness). *Suppose that initial data satisfy the incompressible constraints (1.2) and (1.3), $a_0 \in \tilde{B}_\mu^{\frac{N}{2},\infty}$, $u_0 \in \dot{B}_{2,1}^{\frac{N}{2}-1}$ and $E_0 \in \tilde{B}_\mu^{\frac{N}{2},\infty}$ with*

$$\|a_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + \|E_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} \leq \lambda,$$

where λ is a small positive constant. Then there exists a unique global solution for system (1.2) with

$$a \in C(\mathbb{R}^+; \tilde{B}_\mu^{\frac{N}{2},\infty}), \quad (u, E) \in Y^{\frac{N}{2}} \quad \text{and} \quad \nabla \Pi \in L^1(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{N}{2}-1}).$$

Besides, the following estimate is valid

$$\begin{aligned} \|a\|_{\tilde{L}_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} + \|(u, E)\|_{Y^{\frac{N}{2}}} \\ \leq C(\|a_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + \|E_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}}), \end{aligned}$$

where C is a constant depending only on N and μ .

The remained sections of this paper are structured as follows. In Section 2, we present some basic properties of Besov spaces. In Section 3, we will study some linear models associated to (1.5). In Section 4, the local theory for (1.5) will be studied and the final section is devoted to discuss the global existence and to give the proof of Theorem 1.2.

2 Littlewood-Paley decomposition results

The proof of most of the results presented in the paper requires a dyadic decomposition of Fourier variables, which is called the *Littlewood-Paley decomposition*. The definition of *Littlewood-Paley decomposition* and Besov space were explained explicitly in [1, 3, 4]. Here we state some classical properties for the Besov spaces.

Proposition 2.1. *The following properties hold true:*

1) *Derivatives: we have*

$$C^{-1}\|u\|_{\dot{B}_{p,r}^s} \leq \|\nabla u\|_{\dot{B}_{p,r}^{s-1}} \leq C\|u\|_{\dot{B}_{p,r}^s}.$$

2) *Algebraic property: for $s > 0$, $\dot{B}_{p,r}^s \cap L^\infty$ is an algebra.*

3) *Real interpolation: $\left(\dot{B}_{p,r}^{s_1}, \dot{B}_{p,r}^{s_2}\right)_{\theta, r'} = \dot{B}_{p,r'}^{\theta s_1 + (1-\theta)s_2}$.*

We recall from [10] the following estimates for the product of two functions. Here we only give the results of homogeneous Besov spaces, and the same results also truth for nonhomogeneous Besov spaces.

Proposition 2.2. *The following estimates hold true:*

$$\|uv\|_{\dot{B}_{2,1}^s} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}_{2,1}^s} + \|v\|_{L^\infty} \|u\|_{\dot{B}_{2,1}^s} \quad \text{if } s > 0,$$

$$\|uv\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{N}{2}}} \lesssim \|u\|_{\dot{B}_{2,1}^{s_1}} \|v\|_{\dot{B}_{2,1}^{s_2}} \quad \text{if } s_1, s_2 \leq \frac{N}{2} \quad \text{and } s_1 + s_2 > 0.$$

Let $\Lambda = \sqrt{-\Delta}$. For $s \in \mathbb{R}$, we denote $\Lambda^s z = \mathcal{F}^{-1}(|\xi|^s \hat{z})$, where \hat{z} is the Fourier transform of z . The aim of this paper is to get the global existence of solutions to system (1.5). For this, we define $d^{ij} = -\Lambda^{-1} \partial_j u^i$, then $u^i = \Lambda^{-1} \partial_j d^{ij}$. Applying $-\Lambda^{-1} \partial_j$ to the second equation of system (1.5), we have

$$\partial_t d^{ij} - \mu \Delta d^{ij} + u \cdot \nabla d - \Lambda E_{ij} = H.$$

H will be determined in Section 5. Taking H as a function independent of d and E , combination with the third equation of system (1.5), we have the following linear system

$$\begin{cases} \partial_t E + \Lambda d = R, \\ \partial_t d^{ij} - \mu \Delta d^{ij} - \Lambda E_{ij} = H. \end{cases} \quad (2.1)$$

Using the spectral analysis as in [7], we may expect that system (2.1) has a parabolic smoothing effect on d and on the low frequencies of E , while expect a damping effect on the high frequencies of E . To get the optimal estimates, we need to introduce the hybrid spaces which are defined differently for low and high frequencies. One can see the details in [7].

Definition 2.1. *For $\mu > 0$, $r \in [1, +\infty]$ and $s \in \mathbb{R}$, we denote*

$$\|u\|_{\tilde{B}_\mu^{s,r}} = \sum_{q \in \mathbb{Z}} 2^{qs} \max\{\mu, 2^{-q}\}^{1-\frac{2}{r}} \|\dot{\Delta}_q u\|_{L^2}.$$

Obviously we remark that $\|u\|_{\tilde{B}_\mu^{s,\infty}} \approx \|u\|_{\dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s-1}}$ and $\|u\|_{\tilde{B}_\mu^{s,2}} = \|u\|_{\dot{B}_{2,1}^s}$. Also we need to introduce more accurate results which may be obtained by means of paradifferential calculus. It is introduced first by J. M. Bony in [2]. The paraproduct between f and g is defined by

$$\dot{T}_f g = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta}_q g.$$

And define the remainder

$$\dot{R}(f, g) = \sum_{|q-p| \leq 1} \dot{\Delta}_p f \dot{\Delta}_q g.$$

We have the following so-called homogeneous Bony's decomposition:

$$fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f, g).$$

Now let us recall some estimates in hybrid Besov spaces for the product of two functions which one can see Proposition 5.3 in [7].

Proposition 2.3. *Let $r \in [1, \infty]$ and $s, t \in \mathbb{R}$. There exists a constant C such that*

$$\|\dot{T}_u v\|_{\tilde{B}_\mu^{s+t-\frac{N}{2}, r}} \leq \|u\|_{\tilde{B}_\mu^{s, r}} \|v\|_{\dot{B}_{2,1}^t}, \text{ if } s \leq \min\{1 - \frac{2}{r} + \frac{N}{2}, \frac{N}{2}\},$$

$$\|\dot{T}_u v\|_{\tilde{B}_\mu^{s+t-\frac{N}{2}, r}} \leq \|u\|_{\dot{B}_{2,1}^s} \|v\|_{\tilde{B}_\mu^{t, r}}, \text{ if } s \leq \frac{N}{2},$$

$$\|\dot{R}(u, v)\|_{\tilde{B}_\mu^{s+t-\frac{N}{2}, r}} \leq \|u\|_{\tilde{B}_\mu^{s, r}} \|v\|_{\dot{B}_{2,1}^t}, \text{ if } s+t > \max\{0, 1 - \frac{2}{r}\}.$$

Proposition 2.4. *Let $s, t \in \mathbb{R}$. There exists a constant C such that*

$$\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s+t-\frac{N}{2}}} \leq \|u\|_{\tilde{B}_\mu^{s, \infty}} \|v\|_{\tilde{B}_\mu^{t, 1}}, \text{ if } s \leq \frac{N}{2},$$

$$\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s+t-\frac{N}{2}}} \leq \|u\|_{\tilde{B}_\mu^{s, 1}} \|v\|_{\tilde{B}_\mu^{t, \infty}}, \text{ if } s \leq \frac{N}{2} - 1,$$

$$\|\dot{R}(u, v)\|_{\dot{B}_{2,1}^{s+t-\frac{N}{2}}} \leq \|u\|_{\tilde{B}_\mu^{s, \infty}} \|v\|_{\tilde{B}_\mu^{t, 1}}, \text{ if } s+t > 0.$$

Proof. From the definition of $\dot{T}_u v$, we can write

$$\dot{\Delta}_q \dot{T}_u v = \sum_{|q-q'|\leq 3} \dot{\Delta}_q (\dot{S}_{q'-1} u \dot{\Delta}_{q'} v),$$

whence

$$\begin{aligned} \|\dot{\Delta}_q \dot{T}_u v\|_{L^2} &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} \|\dot{\Delta}_{q''} u\|_{L^\infty} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} 2^{q''\frac{N}{2}} \|\dot{\Delta}_{q''} u\|_{L^2} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} \max\{\mu, 2^{-q''}\} \|\dot{\Delta}_{q''} u\|_{L^2} \cdot \max\{\mu, 2^{-q'}\}^{-1} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\quad \times 2^{q''\frac{N}{2}} \max\{\mu, 2^{-q''}\}^{-1} \cdot \max\{\mu, 2^{-q'}\}. \end{aligned}$$

It is now clear that

$$\max\{\mu, 2^{-q''}\}^{-1} \max\{\mu, 2^{-q'}\} \leq \max\{1, 2^{q''-q'}\} \leq 1.$$

So if $s \leq \frac{N}{2}$, the convolution inequality yields

$$\sum_{q \in \mathbb{Z}} 2^{q(s+t-\frac{N}{2})} \|\dot{\Delta}_q \dot{T}_u v\|_{L^2} \leq C \|u\|_{\tilde{B}_\mu^{s, \infty}} \|v\|_{\tilde{B}_\mu^{t, 1}}.$$

For proving the second result, similarly, we notice that

$$\begin{aligned} \|\dot{\Delta}_q \dot{T}_u v\|_{L^2} &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} \|\dot{\Delta}_{q''} u\|_{L^\infty} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} 2^{q''\frac{N}{2}} \|\dot{\Delta}_{q''} u\|_{L^2} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\leq \sum_{\substack{|q'-q|\leq 4 \\ q''\leq q'-2}} \max\{\mu, 2^{-q''}\}^{-1} \|\dot{\Delta}_{q''} u\|_{L^2} \cdot \max\{\mu, 2^{-q'}\} \|\dot{\Delta}_{q'} v\|_{L^2} \\ &\quad \times 2^{q''\frac{N}{2}} \max\{\mu, 2^{-q''}\} \cdot \max\{\mu, 2^{-q'}\}^{-1}. \end{aligned}$$

For $q'' \leq q' - 2$, we see that

$$\max\{\mu, 2^{-q''}\} \max\{\mu, 2^{-q'}\}^{-1} \leq 2^{q'-q''},$$

then

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(s+t-\frac{N}{2})} \|\dot{\Delta}_q \dot{T}_u v\|_{L^2} &\leq C \sum_{q \in \mathbb{Z}} \sum_{\substack{|q'-q| \leq 4 \\ q'' \leq q'-2}} \max\{\mu, 2^{-q''}\}^{-1} 2^{q''s} \|\dot{\Delta}_{q''} u\|_{L^2} \\ &\quad \times \max\{\mu, 2^{-q'}\} 2^{q't} \|\dot{\Delta}_{q'} v\|_{L^2} 2^{(q-q')t} 2^{(q''-q')(\frac{N}{2}-s-1)}. \end{aligned}$$

The convolution inequality implies

$$\sum_{q \in \mathbb{Z}} 2^{q(s+t-\frac{N}{2})} \|\dot{\Delta}_q \dot{T}_u v\|_{L^2} \leq C \|u\|_{\tilde{B}_\mu^{s,1}} \|v\|_{\tilde{B}_\mu^{t,\infty}},$$

if $s \leq \frac{N}{2} - 1$.

To prove the result on $\dot{R}(u, v)$, we note that

$$\dot{\Delta}_q \dot{R}(u, v) = \sum_{q' \geq q-2} (\dot{\Delta}_{q'} u \tilde{\Delta}_{q'} v).$$

This entails

$$\begin{aligned} \|\dot{\Delta}_q \dot{R}(u, v)\|_{L^2} &\leq 2^{q\frac{N}{2}} \sum_{q \leq q'+2} \|\dot{\Delta}_{q'} u\|_{L^2} \|\tilde{\Delta}_{q'} v\|_{L^2} \\ &\leq 2^{q\frac{N}{2}} \sum_{q \leq q'+2} \max\{\mu, 2^{-q'}\} \|\dot{\Delta}_{q'} u\|_{L^2} \min\{\mu^{-1}, 2^{q'}\} \|\dot{\Delta}_{q'} v\|_{L^2}. \end{aligned}$$

If $s + t > 0$, then convolution inequality yields

$$\sum_{q \in \mathbb{Z}} 2^{q(s+t-\frac{N}{2})} \|\dot{\Delta}_q \dot{R}(u, v)\|_{L^2} \leq C \|u\|_{\tilde{B}_\mu^{s,\infty}} \|v\|_{\tilde{B}_\mu^{t,1}}.$$

□

We will also use the so called Chemin-Lerner type spaces $\tilde{L}_T^\rho(B_{p,r}^s)$ which are described in detail in [5]. The case of nonhomogeneous Besov space can be defined in the same way.

Definition 2.2. For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$, and $T \in (0, +\infty]$, we get

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{2,r}^s)} = \left(\sum_{q \in \mathbb{Z}} 2^{qs} \left(\int_0^T \|\dot{\Delta}_q u(t)\|_{L^2}^\rho dt \right)^{\frac{\rho}{\rho-1}} \right)^{\frac{1}{\rho}}$$

and denote by $\tilde{L}_T^\rho(\dot{B}_{2,r}^s)$ the subset of distributions $u \in \mathcal{S}'(0, T) \times \mathcal{S}'_h(\mathbb{R}^N)$ with finite $\|u\|_{\tilde{L}_T^\rho(\dot{B}_{2,r}^s)}$ norm.

When $T = +\infty$, the index T is omitted. We further denote $\tilde{C}_T(\dot{B}_{2,r}^s) = C([0, T]; \dot{B}_{2,r}^s) \cap \tilde{L}_T^\infty(\dot{B}_{2,r}^s)$ and $\tilde{L}_T^\rho(\dot{B}_{2,r}^s \cap L^\infty) = \tilde{L}_T^\rho(\dot{B}_{2,r}^s) \cap L_T^\rho(L^\infty)$.

3 The linearized equations

3.1 The transport equation

Here, we present a priori estimate for the linear transport equation which has been stated in [1] (Theorem 3.14).

Proposition 3.1. *Let $1 \leq p \leq p_1 \leq \infty$, $r \in [1, +\infty]$, $T > 0$ and $\frac{1}{p'} := 1 - \frac{1}{p}$. Assume that*

$$\begin{cases} s > -N \min(\frac{1}{p_1}, \frac{1}{p'}), & \text{if } \operatorname{div} u \neq 0, \\ s > -1 - N \min(\frac{1}{p_1}, \frac{1}{p'}), & \text{if } \operatorname{div} u = 0. \end{cases}$$

Suppose $a_0 \in B_{p,r}^s$, $g \in L^1(0, T; B_{p,r}^s)$ and that $a \in L^\infty(0, T; B_{p,r}^s)$ solves

$$\begin{cases} \partial_t a + \operatorname{div}(ua) = g, \\ a|_{t=0} = a_0. \end{cases}$$

Then there exists a constant C depending only on s, p, p_1, r such that the following inequality holds, if $t \in [0, T]$,

$$\|a\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq e^{CV(t)} \left(\|a_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|g(\tau)\|_{B_{p,r}^s} d\tau \right),$$

$$\text{with } \begin{cases} V(t) = \int_0^t \|\nabla u(\tau)\|_{B_{p_1, r \cap L^\infty}^{\frac{N}{2}}} d\tau, & \text{if } s < 1 + \frac{N}{p_1}, \\ V(t) = \int_0^t \|\nabla u(\tau)\|_{B_{p_1, r}^{s-1}} d\tau, & \text{if } s > 1 + \frac{N}{p_1} \text{ or } s = 1 + \frac{N}{p_1}, r = 1. \end{cases}$$

3.2 The linearized momentum equation

When the density is close to a constant, we are led to study the following linearized momentum equations:

$$\begin{cases} \partial_t u + v \cdot \nabla u - \mu b \Delta u + b \nabla \Pi = f, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (3.1)$$

where $b := a + 1$ is bounded below by a positive constant \underline{b} . That is $\inf_{x \in \mathbb{R}^N} b(x) \geq \underline{b}$. Before stating our result, let us introduce the following notation:

$$A_T = 1 + \underline{b} 2^{N_0 \alpha} \|\nabla b\|_{B_{2,1}^{\frac{N}{2}-1}}$$

for $\alpha \in (0, 1)$.

Proposition 3.2. *Let $s \in (1 - \frac{N}{2}, 1 + \frac{N}{2})$ and $0 < \alpha < 1$. Also we assume $\alpha < \frac{s-1}{2}$ if $s > 1$ and $a_0 \in B_{2,1}^{\frac{N}{2}}$. Let u_0 be a divergence-free vector field with coefficients in $B_{2,r}^{s-1}$ for $r \in [1, \infty]$, and f be a time-dependent vector field with coefficients in $\tilde{L}_T^1(B_{2,r}^{s-1})$. u, v are two divergence-free time-dependent vector fields such that $\nabla v \in L^1(0, T; B_{2,1}^{\frac{N}{2}})$ and $u \in \tilde{C}([0, T]; B_{2,r}^{s-1}) \cap \tilde{L}_T^1(B_{2,r}^{s+1})$. In addition, assume that (3.1) is fulfilled for some distribution Π . Let N_0 be a positive integer such that $b_{N_0} = 1 + S_{N_0} a$ satisfies*

$$\inf_{x \in \mathbb{R}^N} b_{N_0} \geq \frac{1}{2} \underline{b}.$$

Denoting $\underline{\mu} := \mu \inf_{x \in \mathbb{R}^N} (a + 1)$, then there exists a constant $C = C(s, N, \mu, \underline{\mu})$ such that if additionally,

$$C A_T^{\kappa+1} \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \min\left\{\frac{1}{4} \underline{b}, \frac{1}{4\mu} \underline{\mu}\right\},$$

the following estimate holds for $k = \frac{|s-1|}{\alpha}$,

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(B_{2,r}^{s-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} + \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\ & \leq C e^{CV(T)} \left(\|u_0\|_{B_{2,r}^{s-1}} + A_T^k (\|f\|_{L_T^1(B_{2,1}^{s-1})} + \mu A_T \|u\|_{L_T^1(B_{2,r}^{s+1-\alpha})}) \right), \end{aligned} \quad (3.2)$$

with $V(t) = \int_0^t (\|\nabla v(\tau)\|_{B_{2,1}^{\frac{N}{2}}} + 2^{2N_0} \|a\|_{B_{2,1}^{\frac{N}{2}}}) d\tau$.

Proof. For positive integer N_0 , we rewrite (3.1) as

$$\begin{cases} \partial_t u + v \cdot \nabla u - \mu b_{N_0} \Delta u + b \nabla \Pi = f + E_{N_0}, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (3.3)$$

with $E_{N_0} = \mu(a - S_{N_0} a) \Delta u$ and $b_{N_0} = 1 + S_{N_0} a$.

Applying the operator Δ_q to (3.3), denoting $u_q = \Delta_q u$ and $\Pi_q = \Delta_q \Pi$, then we have

$$\begin{aligned} & \partial_t u_q + v \cdot \nabla u_q - \mu \operatorname{div}(b_{N_0} \nabla u_q) + \nabla \Pi_q \\ &= f_q - \Delta_q(a \nabla \Pi) + [v, \Delta_q] \cdot \nabla u + R_q + \Delta_q E_{N_0}, \end{aligned}$$

where $R_q := \mu \Delta_q(S_{N_0} a \Delta u) - \mu \operatorname{div}(S_{N_0} a \nabla u_q)$. Multiplying this equation by u_q , integrating by parts over \mathbb{R}^N yields

$$\begin{aligned} \frac{d}{dt} \|u_q\|_{L^2}^2 + \underline{\mu} \|\nabla u_q\|_{L^2}^2 &\leq C \|u_q\|_{L^2} (\|f_q\|_{L^2} + \|[v, \Delta_q] \cdot \nabla u\|_{L^2} \\ &\quad + \|R_q\|_{L^2} + \|\Delta_q E_{N_0}\|_{L^2}) - 2 \int_{\mathbb{R}^N} \Delta_q(a \nabla \Pi) u_q dx, \end{aligned} \quad (3.4)$$

with $\underline{\mu} := \mu \underline{b}$.

Denoting $\tilde{a} := a - \Delta_{-1} a$, since $\operatorname{div} u_q = 0$, the last term of (3.4) can be written as

$$2 \int_{\mathbb{R}^N} \Delta_q(a \nabla \Pi) u_q dx = 2 \int_{\mathbb{R}^N} \Delta_q(-T_{\nabla \tilde{a}} \Pi + T_{\nabla \Pi} a + R(a, \nabla \Pi) + T_{\Delta_{-1} \nabla a} \Pi) u_q dx$$

by integration by parts and Bony's decomposition. According to Bernstein inequality, there exists a $\kappa_0 > 0$ such that for all $q \geq 0$, we have $\kappa_0 2^q \|\Delta_q u\|_{L^2} \leq \|\Delta_q \nabla u\|_{L^2}$. Integrating over $[0, T]$, (3.4) implies

$$\begin{aligned} & \|u_q\|_{L^2} + \underline{\mu} 2^{2q} \int_0^T \|u_q\|_{L^2} dt \\ &\leq C \|\Delta_q u_0\|_{L^2} + C \int_0^T \left(\|f_q\|_{L^2} + \|\Delta_q E_{N_0}\|_{L^2} + \underline{\mu} \delta_{-1, q} \|\Delta_{-1} u\|_{L^2} \right. \\ &\quad + \|[v, \Delta_q] \cdot \nabla u\|_{L^2} + \|R_q\|_{L^2} + \|\Delta_q(T_{\nabla \tilde{a}} \Pi)\|_{L^2} + \|\Delta_q(T_{\Delta_{-1} \nabla a} \Pi)\|_{L^2} \\ &\quad \left. + \|\Delta_q(T_{\nabla \Pi} a)\|_{L^2} + \|\Delta_q R(a, \nabla \Pi)\|_{L^2} \right) dt \end{aligned}$$

for all $q \geq -1$. Elementary computations yield

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(B_{2,r}^{s-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} \\ &\leq C \|u_0\|_{B_{2,r}^{s-1}} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} + \left(\sum_{q \in \mathbb{Z}} 2^{qr(s-1)} \int_0^T \left(\|f_q\|_{L^2} \right. \right. \\ &\quad + \|R_q\|_{L^2} + \|[v, \Delta_q] \cdot \nabla u\|_{L^2} + \|\Delta_q(T_{\nabla \tilde{a}} \Pi)\|_{L^2} + \|\Delta_q(T_{\nabla \Pi} a)\|_{L^2} \\ &\quad \left. \left. + \|\Delta_q R(a, \nabla \Pi)\|_{L^2} + \|\Delta_q E_{N_0}\|_{L^2} + \|\Delta_q(T_{\Delta_{-1} \nabla a} \Pi)\|_{L^2} \right)^r dt \right)^{\frac{1}{r}} \\ &\leq C \|u_0\|_{B_{2,r}^{s-1}} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} + \|f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \sum_{i=1}^7 I_i. \end{aligned} \quad (3.5)$$

Now, we estimate the series of the right hand side of (3.5) term by term.

As $1 - \frac{N}{2} < s < 1 + \frac{N}{2}$, we can use Lemma B.2 in [11] to bound I_2 . Indeed,

$$\begin{aligned} I_2 &= \left(\sum_{q \geq -1} 2^{qr(s-1)} \int_0^T \|[v, \Delta_q] \cdot \nabla u\|_{L^2}^r dt \right)^{\frac{1}{r}} \\ &\leq C \int_0^T \|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s-1}} dt. \end{aligned} \quad (3.6)$$

By Lemma B.3 in [11], the following estimate

$$\begin{aligned}
I_1 &= \left(\sum_{q \geq -1} 2^{qr(s-1)} \int_0^T \|R_q\|_{L^2}^2 dt \right)^{\frac{1}{r}} \\
&\leq C \mu \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}+\alpha})} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} \\
&\leq C \mu 2^{N_0 \alpha} \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})}
\end{aligned} \tag{3.7}$$

is proved.

By the definition of Bony decomposition, we see that for any two functions f and g

$$\begin{aligned}
2^{q(s-1)} \|T_{\nabla f} g\|_{L^2} &\leq C \sum_{\substack{|q'-q| \leq 4 \\ q'' \leq q'-2}} 2^{q(s-1)} \|\Delta_{q''} \nabla f\|_{L^2} 2^{q'' \frac{N}{2}} \|\Delta_{q'} g\|_{L^2} \\
&\leq C \sum_{\substack{|q'-q| \leq 4 \\ q'' \leq q'-2}} \|\Delta_{q''} \nabla f\|_{L^2} 2^{q''(\frac{N}{2}+\alpha-1)} \cdot 2^{q'(s-1-\alpha)} \|\Delta_{q'} \nabla g\|_{L^2} \\
&\quad \times 2^{(q-q')(s-1)} 2^{(q''-q')(1-\alpha)}.
\end{aligned}$$

Thus by the convolution inequality, I_3 is estimated by

$$\begin{aligned}
I_3 &\leq C \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\
&\quad + C \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,r}^{\frac{N}{2}+\alpha})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})},
\end{aligned}$$

where we have used the above estimate with $f = S_{N_0} \tilde{a}$ and $g = \Pi$.

Form Proposition 2.2, Theorem 2.82 and Theorem 2.85 in [1], $I_4 + I_5$ is bounded by

$$\begin{aligned}
I_4 + I_5 &\leq C \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\
&\quad + C \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}+\alpha})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})}.
\end{aligned} \tag{3.8}$$

As to I_6 , also the standard continuity result for para-product implies

$$I_6 \leq C \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})}. \tag{3.9}$$

Obviously, Theorem 2.82 and Theorem 2.85 in [1], implies

$$\begin{aligned}
I_7 &\leq C \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\
&\quad + C \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}+\alpha})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})}.
\end{aligned}$$

Thus, combining the above estimates for I_1 to I_6 , we obtain

$$\begin{aligned}
&\|u\|_{\tilde{L}_T^\infty(B_{2,r}^{s-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} \\
&\lesssim \|u_0\|_{B_{2,r}^{s-1}} + \|f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \mu \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} \\
&\quad + \int_0^T \left(\|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s-1}} + \mu 2^{N_0 \alpha} \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|u\|_{B_{2,r}^{s+1-\alpha}} \right) dt \\
&\quad + \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} \\
&\quad + \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}+\alpha})} \|\nabla \Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})}.
\end{aligned} \tag{3.10}$$

On the other hand, $\nabla \Pi$ solves the following elliptic equation:

$$\operatorname{div}(b_{N_0} \nabla \Pi) = \operatorname{div} L - F_{N_0},$$

with $L = f + \mu a \Delta u - v \cdot \nabla u$ and $F_{N_0} = \operatorname{div}((a - S_{N_0} a) \nabla \Pi)$. Apply Δ_q to the above equation we get

$$\operatorname{div}(b_{N_0} \nabla \Pi_q) = \operatorname{div} L_q - \Delta_q F_{N_0} + \tilde{R}_q, \quad (3.11)$$

with $\tilde{R}_q = \operatorname{div}(b_{N_0} \nabla \Pi_q) - \Delta_q \operatorname{div}(b_{N_0} \nabla \Pi)$. Multiplying (3.11) by Π_q and integrating by parts, we obtain

$$\underline{b} \|\nabla \Pi_q\|_{L^2}^2 \leq \left(\|\operatorname{div} L_q\|_{L^2} + \|\Delta_q F_{N_0}\|_{L^2} + \|\tilde{R}_q\|_{L^2} \right) \|\Pi_q\|_{L^2}. \quad (3.12)$$

Bernstein inequality implies that

$$\underline{b} 2^q \|\nabla \Pi_q\|_{L^2} \leq C \left(\|\operatorname{div} L_q\|_{L^2} + \|\Delta_q F_{N_0}\|_{L^2} + \|\tilde{R}_q\|_{L^2} + \underline{b} \|\Delta_{-1} \Pi\| \right).$$

For $-\frac{N}{2} < \sigma \leq \frac{N}{2}$, the second term can be estimated by

$$\sum_{q \geq -1} 2^{q(\sigma-1)} \|\Delta_q F_{N_0}\|_{L^2} \leq C \|a - S_{N_0} a\|_{B_{2,1}^{\frac{N}{2}}} \|\nabla \Pi\|_{B_{2,r}^\sigma}.$$

Then from the assumption,

$$C \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \frac{1}{4} \underline{b}$$

and Lemma B.1 in [11], for $\alpha \in (0, 1)$, we have

$$\underline{b} \|\nabla \Pi\|_{B_{2,r}^\sigma} \lesssim \|\mathcal{Q}L\|_{B_{2,r}^\sigma} + \|S_{N_0} a\|_{B_{2,1}^{\frac{N}{2}+\alpha}} \|\nabla \Pi\|_{B_{2,r}^{\sigma-\alpha}} + \underline{b} \|\nabla \Pi\|_{B_{2,r}^{\sigma-\alpha}}, \quad (3.13)$$

where

$$\mathcal{Q} = \nabla(-\Delta)^{-1} \operatorname{div}, \quad \underline{b} = \mu \inf_{x \in \mathbb{R}^N} (S_{N_0} a + 1) \quad \text{and} \quad -\frac{N}{2} < \sigma \leq \frac{N}{2}.$$

If σ satisfies that $\alpha < \sigma \leq \frac{N}{2}$, by interpolation, we get that

$$\begin{aligned} \underline{b} \|\nabla \Pi\|_{B_{2,r}^\sigma} &\lesssim \|\mathcal{Q}L\|_{B_{2,r}^\sigma} + (\|S_{N_0} a\|_{B_{2,1}^{\frac{N}{2}+\alpha}} + \underline{b}) \|\nabla \Pi\|_{B_{2,r}^{\sigma-\alpha}} \\ &\lesssim \|\mathcal{Q}L\|_{B_{2,r}^\sigma} + (2^{N_0 \alpha} \|S_{N_0} a\|_{B_{2,1}^{\frac{N}{2}}} + \underline{b}) \|\nabla \Pi\|_{B_{2,r}^{\frac{\sigma-\alpha}{\sigma}}} \|\nabla \Pi\|_{L^2}^{\frac{\alpha}{\sigma}}. \end{aligned} \quad (3.14)$$

We conclude by Young's inequality and the L^2 estimate for the pressure in Proposition A.1 in [11] that

$$\underline{b} \|\nabla \Pi\|_{B_{2,r}^\sigma} \lesssim A_T^{\frac{\alpha}{\sigma}} \|\mathcal{Q}L\|_{B_{2,r}^\sigma}. \quad (3.15)$$

Similarly, if σ satisfies that $-\frac{N}{2} < \sigma < -\alpha$,

$$\begin{aligned} \underline{b} \|\nabla \Pi\|_{B_{2,r}^\sigma} &\lesssim \|\mathcal{Q}L\|_{B_{2,r}^\sigma} + (\|S_{N_0} a\|_{B_{2,1}^{\frac{N}{2}}} + \underline{b}) \|\nabla \Pi\|_{B_{2,r}^{\sigma+\alpha}} \\ &\lesssim \|\mathcal{Q}L\|_{B_{2,r}^\sigma} + (\|S_{N_0} a\|_{B_{2,r}^{\frac{N}{2}}} + \underline{b}) \|\nabla \Pi\|_{B_{2,r}^{\frac{\sigma+\alpha}{\sigma}}} \|\nabla \Pi\|_{L^2}^{-\frac{\alpha}{\sigma}}. \end{aligned}$$

Together with (3.15), we can conclude that

$$\underline{b} \|\nabla \Pi\|_{B_{2,r}^\sigma} \lesssim A_T^{\frac{|\sigma|}{\alpha}} \|\mathcal{Q}L\|_{B_{2,r}^\sigma},$$

if $\alpha < \sigma \leq \frac{N}{2}$ or $-\frac{N}{2} < \sigma < -\alpha$. Therefore, we are led to estimate $\mathcal{Q}L$ in $L_T^1(B_{2,1}^{s-1})$. Since $\alpha < \frac{s-1}{2}$, this may be done by making use of Bony's decomposition, Lemma B.2 in [11] and Propositions 2.2,

$$\begin{aligned} \|\mathcal{Q}L\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})} &\lesssim \|\mathcal{Q}f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \int_0^T \|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s-1}} dt \\ &\quad + \mu \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\Delta u\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\ &\quad + \mu \|S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\Delta u\|_{\tilde{L}_T^1(B_{2,r}^{s-1-\alpha})}. \end{aligned} \quad (3.16)$$

Here we have used the fact that $B_{2,1}^{s-1} \hookrightarrow B_{2,1}^{s-1-\alpha}$. Plus (3.16) into (3.10) yields,

$$\begin{aligned}
& \|u\|_{\tilde{L}_T^\infty(B_{2,r}^{s-1})} + \underline{\mu}\|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} + \|\nabla\Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} \\
& \lesssim \|u_0\|_{B_{2,r}^{s-1}} + \|f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \mu A_T^{\kappa+1}\|a - S_{N_0}a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})}\|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} \\
& + \int_0^T \left(\|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s-1}} + \mu 2^{N_0\alpha} \|a\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s+1-\alpha}} \right) dt \\
& + A_T^\kappa \left(\|\mathcal{Q}f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \mu 2^{N_0\alpha} A_T \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} \right) \\
& + \int_0^T \|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s-1}} dt.
\end{aligned} \tag{3.17}$$

The fact that

$$C\mu A_T^{\kappa+1}\|a - S_{N_0}a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \frac{1}{4}\underline{\mu}$$

and interpolation inequality imply that

$$\mu 2^{N_0\alpha} \|a\|_{B_{2,1}^{\frac{N}{2}}} \|u\|_{B_{2,r}^{s+1-\alpha}} \leq C 2^{2N_0} \|a\|_{B_{2,1}^{\frac{N}{2}}}^{\frac{2}{\alpha}} \|u\|_{B_{2,r}^{s-1}} + \frac{1}{4}\underline{\mu}\|u\|_{B_{2,r}^{s+1}}.$$

Let $X(t) = \|u\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} + \underline{\mu}\|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1})} + \|\nabla\Pi\|_{\tilde{L}_T^1(B_{2,r}^{s-1})}$, from the above estimate, we get

$$\begin{aligned}
X(t) & \lesssim \|u_0\|_{B_{2,r}^{s-1}} + A_T^k \left(\|f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \mu 2^{N_0\alpha} A_T \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} \right) \\
& + \int_0^T \left(\|\nabla v\|_{B_{2,1}^{\frac{N}{2}}} + 2^{2N_0} \|a\|_{B_{2,1}^{\frac{N}{2}}}^{\frac{2}{\alpha}} \right) X(t) dt.
\end{aligned} \tag{3.18}$$

Then Gronwall lemma yields

$$\begin{aligned}
X(T) & \leq C e^{CV(T)} \left(\|u_0\|_{B_{2,r}^{s-1}} \right. \\
& \left. + A_T^k \|f\|_{\tilde{L}_T^1(B_{2,r}^{s-1})} + \mu 2^{N_0\alpha} A_T^{k+1} \|u\|_{\tilde{L}_T^1(B_{2,r}^{s+1-\alpha})} \right).
\end{aligned} \tag{3.19}$$

□

Now we turn to give the priori estimate of the mixed linear system

$$\begin{cases} \partial_t E + u \cdot \nabla E + \Lambda d = F, \\ \partial_t d + u \cdot \nabla d - \mu \Delta d - \Lambda E = G. \end{cases} \tag{3.20}$$

We have the following proposition which one can see the details of proof in [7].

Proposition 3.3. *Let (E, d) be a solution of (3.20) on $[0, T]$ with initial data (E_0, d_0) , $1 - \frac{N}{2} < s \leq 1 + \frac{N}{2}$ and $V(t) = \int_0^t \|\nabla u(\tau)\|_{\dot{B}_{2,1}^{\frac{N}{2}}} d\tau$. Then the following estimate holds*

$$\begin{aligned}
& \|E(t)\|_{\tilde{B}_\mu^{s,\infty}} + \|d(t)\|_{\dot{B}_{2,1}^{s-1}} + \mu \int_0^T \left(\|E(\tau)\|_{\tilde{B}_\mu^{s,1}} + \|d(\tau)\|_{\dot{B}_{2,1}^{s+1}} \right) d\tau \leq C e^{CV(t)} \\
& \times \left(\|E_0\|_{\tilde{B}_\mu^{s,\infty}} + \|d_0\|_{\dot{B}_{2,1}^{s-1}} + \mu \int_0^T e^{-CV(\tau)} (\|F(\tau)\|_{\tilde{B}_\mu^{s,\infty}} + \|G(\tau)\|_{\dot{B}_{2,1}^{s-1}}) d\tau \right).
\end{aligned}$$

4 Local well-posedness for data with critical regularity

In this section, we will obtain the local existence of solutions to system (1.5). We proceed by the following steps.

4.1 A priori estimates

Let $(u_L, \nabla \Pi_L)$ solves the non-stationary Stokes system

$$\begin{cases} \partial_t u_L - \mu \Delta u_L + \nabla \Pi_L = 0, \\ \operatorname{div} u_L = 0, \\ u_L|_{t=0} = u_0. \end{cases} \quad (4.1)$$

It is easy to obtain that $u_L \in C([0, T]; B_{2,1}^{\frac{N}{2}-1}) \cap L^1(0, T; B_{2,1}^{\frac{N}{2}+1})$ and $\nabla \Pi_L \in L^1(0, T; B_{2,1}^{\frac{N}{2}-1})$. Assume that T has been chosen so small as to satisfy

$$\|u_L\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} \leq \|u_0\|_{B_{2,1}^{\frac{N}{2}-1}}, \quad (4.2)$$

and

$$\mu \|u_L\|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + \|\nabla \Pi_L\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \leq \lambda, \quad (4.3)$$

where λ will be determined later.

Let $\bar{u} = u - u_L$, $\nabla \bar{\Pi} = \nabla \Pi - \nabla \Pi_L$, where $(a, u, \nabla \Pi)$ satisfies (1.5) on $[0, T] \times \mathbb{R}^N$. Suppose that $a \in C^1([0, T]; B_{2,1}^{\frac{N}{2}})$, $u \in C^1([0, T]; B_{2,1}^{\frac{N}{2}}) \cap L_T^1(B_{2,1}^{\frac{N}{2}+1})$ and $\nabla \Pi \in L^1(0, T; B_{2,1}^{\frac{N}{2}-1})$. We can deduce that $(a, \bar{u}, \nabla \bar{\Pi}, E)$ satisfies the following system

$$\begin{cases} \partial_t a + (\bar{u} + u_L) \cdot \nabla a = 0, \\ \partial_t \bar{u} + (\bar{u} + u_L) \cdot \nabla \bar{u} - \mu(a+1)\Delta \bar{u} + (a+1)\nabla \bar{\Pi} = F + G, \\ \partial_t E + \bar{u} \cdot \nabla E = H, \\ \operatorname{div} \bar{u} = 0, \\ (a, \bar{u}, E)|_{t=0} = (a_0, 0, E_0), \end{cases} \quad (4.4)$$

where

$$\begin{aligned} F &= -(\bar{u} + u_L) \nabla u_L + \mu a \Delta u_L - a \nabla \Pi_L, \\ G &= (a+1)(\partial_j E_{ik} E_{jk} + \partial_j E_{ij}), \\ H &= -u_L \nabla E + (\nabla \bar{u} + \nabla u_L) E + \nabla \bar{u} + \nabla u_L. \end{aligned}$$

Denote $U_0 := \|u_0\|_{B_{2,1}^{\frac{N}{2}-1}}$, $U_L := \mu \|u_L\|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + \|\nabla \Pi_L\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})}$. Assume that the following inequalities are fulfilled for some suitable λ , \tilde{U}_0 and T :

$$\begin{cases} \|a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq 2\|a_0\|_{B_{2,1}^{\frac{N}{2}}}, \\ A_T^{\kappa+1} \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \min\{\frac{1}{4C}b, \frac{1}{4C\mu}\underline{\mu}\}, \\ \|E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq 6\|E_0\|_{B_{2,1}^{\frac{N}{2}}}, \\ \|\bar{u}\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} + \underline{\mu} \|\bar{u}\|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + \|\nabla \bar{\Pi}\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \leq \lambda \tilde{U}_0. \end{cases} \quad (4.5)$$

Then we are going to prove that they are actually satisfied with strict inequalities. Since (4.5) depend continuously on the time variable and are satisfied with strict inequalities initially, a basic bootstrap argument insures that (4.5) are indeed satisfied for small T . For convenience, we denote

$$\overline{U}(T) = \|\bar{u}\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} + \underline{\mu} \|\bar{u}\|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + \|\nabla \bar{\Pi}\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})}.$$

First we prove (4.5)₁ holds with strict inequality. Form Propositions 3.1, we easily obtain that

$$\|a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq e^{C\tilde{V}(T)} \|a_0\|_{B_{2,1}^{\frac{N}{2}}} \leq e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} \|a_0\|_{B_{2,1}^{\frac{N}{2}}}, \quad (4.6)$$

where

$$\tilde{V}(T) = \int_0^T (\|\nabla \bar{u}\|_{B_{2,1}^{\frac{N}{2}}} + \|\nabla u_L\|_{B_{2,1}^{\frac{N}{2}}}) dt.$$

If we choose λ small enough such that

$$e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} < 2, \quad (4.7)$$

then (4.5)₁ holds with strict inequality on $[0, T)$.

Similarly, from Propositions 3.1, we have

$$\begin{aligned} \|E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} &\leq e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} \left(\|E_0\|_{B_{2,1}^{\frac{N}{2}}} + \int_0^T \|(\nabla \bar{u} + \nabla u_L)(E + I)\|_{B_{2,1}^{\frac{N}{2}}} dt \right) \\ &\leq e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} \left(\|E_0\|_{B_{2,1}^{\frac{N}{2}}} + \|(\nabla \bar{u}, \nabla u_L)\|_{L_T^1(B_{2,1}^{\frac{N}{2}})} (\|E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1) \right) \\ &\leq e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} \left(\|E_0\|_{B_{2,1}^{\frac{N}{2}}} + \left(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 \right) (\|E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1) \right) \\ &< 6 \|E_0\|_{B_{2,1}^{\frac{N}{2}}}, \end{aligned}$$

when λ_1 satisfies

$$\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 < \frac{1}{8}, \quad e^{C(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0)} < 2, \quad \frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 < \frac{1}{8} \|E_0\|_{B_{2,1}^{\frac{N}{2}}}. \quad (4.8)$$

Thus (4.5)₃ holds with strict inequality.

According to the estimate (3.14) in [1] (page 134), we get that

$$\|\Delta_j a\|_{L_T^\infty(L^2)} \leq \|\Delta_j a_0\|_{L^2} + C c_j 2^{-j \frac{N}{2}} A_0 \|\nabla(\bar{u} + u_L)\|_{L_T^1(B_{2,1}^{\frac{N}{2}})},$$

where the l^1 norm of c_j equals to 1 and $A_0 = 1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}}$. By the definition of Besov norm, we see that

$$\begin{aligned} \|a - S_{N_0} a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} &= \sum_{q \geq -1} 2^{q \frac{N}{2}} \|\Delta_q(a - S_{N_0} a)\|_{L_T^\infty(L^2)} \\ &\leq \sum_{j=N_0}^{\infty} \sum_{|q-j| \leq 2} 2^{q \frac{N}{2}} \|\Delta_q \Delta_j a\|_{L_T^\infty(L^2)} \\ &\leq C \sum_{j=N_0}^{\infty} \sum_{|q-j| \leq 2} 2^{(q-j) \frac{N}{2}} \left(2^{j \frac{N}{2}} \|\Delta_j a_0\|_{L^2} + A_0 \left(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 \right) c_j \right) \\ &\leq C \sum_{j=N_0}^{\infty} 2^{j \frac{N}{2}} \|\Delta_j a_0\|_{L^2} + C A_0 \left(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 \right). \end{aligned}$$

Since $a_0 \in B_{2,1}^{\frac{N}{2}}$ and $A_T \leq 2A_0$, we can select N_0 large enough such that

$$C \sum_{j=N_0}^{\infty} 2^{j \frac{N}{2}} \|\Delta_j a_0\|_{L^2} < (2A_0)^{-(\kappa+1)} \min\left\{ \frac{1}{16} \underline{b}, \frac{1}{16\mu} \underline{\mu} \right\}.$$

So (4.5)₂ holds with strict inequality provide

$$C A_0 \left(\frac{\lambda}{\mu} + \frac{\lambda}{\underline{\mu}} \tilde{U}_0 \right) < (2A_0)^{-(\kappa+1)} \min\left\{ \frac{1}{16} \underline{b}, \frac{1}{16\mu} \underline{\mu} \right\}. \quad (4.9)$$

Finally, we set T small enough such that

$$4C 2^{2N_0} \|a_0\|_{B_{2,1}^{\frac{N}{2}}}^{\frac{2}{\alpha}} T \leq \log 2,$$

which combination with (4.8) implies

$$e^{CV(T)} \leq 4,$$

where $V(T)$ is defined in Proposition 3.2. Thus from Proposition 3.2, we obtain that

$$\begin{aligned} \overline{U}(T) &\leq C e^{CV(T)} A_T^\kappa \left(\| -(\bar{u} + u_L) \nabla u_L + \mu a \Delta u_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \right. \\ &\quad + \| a \nabla \Pi_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} + \| (a+1) \partial_j E_{ik} E_{jk} \|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \\ &\quad \left. + \| (a+1) \partial_j E_{ij} \|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} + \mu A_T \| \bar{u} \|_{L_T^1(B_{2,1}^{\frac{N}{2}+1-\alpha})} \right) \\ &\leq 4C A_T^\kappa \left(\overline{U}(T) (\| \nabla u_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}})} + U_0 \| \nabla u_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}})} + T \| E \|^2_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \right. \\ &\quad + \mu \| a \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \| u_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + T \| a \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \| E \|^2_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \\ &\quad + T \| a \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \left(2 \| E \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1 \right) + T \| E \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \\ &\quad + \| a \|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \| \nabla \Pi_L \|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} + 2C \mu A_T^{\kappa+1} T^{\frac{\alpha}{2}} \| \bar{u} \|_{\tilde{L}_T^1(B_{2,1}^{\frac{N}{2}+1})} \\ &\quad \left. + \frac{1}{8C A_T^\kappa} \| \bar{u} \|_{L_T^\infty(B_{2,1}^{\frac{N}{2}-1})} \right). \end{aligned} \quad (4.10)$$

If we assume

$$\lambda \leq \frac{1}{16C} \mu, \quad (4.11)$$

then we have

$$\begin{aligned} \overline{U}(t) &\leq 16C \left(\left(\frac{1}{\mu} U_0 + 2 \frac{1}{\mu} \| a_0 \|_{B_{2,1}^{\frac{N}{2}}} \right) + 2 \| a_0 \|_{B_{2,1}^{\frac{N}{2}}} \right) \lambda + 36T \| E_0 \|^2_{B_{2,1}^{\frac{N}{2}}} \\ &\quad + 144T \| a_0 \|_{B_{2,1}^{\frac{N}{2}}} \| E_0 \|^2_{B_{2,1}^{\frac{N}{2}}} + 48T \| a_0 \|_{B_{2,1}^{\frac{N}{2}}} \| E_0 \|_{B_{2,1}^{\frac{N}{2}}} + 6T \| E_0 \|_{B_{2,1}^{\frac{N}{2}}} \\ &\quad + 2T \| a_0 \|_{B_{2,1}^{\frac{N}{2}}} \Big) + 64C 2^{N_0 \alpha} \mu \frac{1}{\underline{\mu}} (1 + \| a_0 \|_{B_{2,1}^{\frac{N}{2}}})^{\kappa+1} T^{\frac{\alpha}{2}} \lambda \tilde{U}_0 \\ &\leq C_0 (U_0 + 1) \lambda + C_0 T + C_0 2^{N_0 \alpha} T \lambda \tilde{U}_0, \end{aligned} \quad (4.12)$$

where C_0 is a general constant depending only on $\| a_0 \|_{B_{2,1}^{\frac{N}{2}}}$, $\| a_0 \|_{B_{2,1}^{\frac{N}{2}}}$, μ , $\underline{\mu}$. Hence, selecting $\tilde{U}_0 = 8C_0(U_0 + 1)$, for fixed λ which determined by (4.7), (4.8), (4.9) and (4.11), we can choose T small enough such that

$$C_0 2^{N_0 \alpha} T < \frac{1}{4}, \quad 4C 2^{2N_0} \| a_0 \|_{B_{2,1}^{\frac{N}{2}}}^{\frac{2}{\alpha}} T \leq \log 2, \quad C_0 T < \frac{1}{8} \lambda \tilde{U}_0. \quad (4.13)$$

This implies (4.5)₄ holds with strict inequality.

4.2 Friedrichs Approximation and uniform estimates

Let L_n^2 be the set of functions spectrally supported in the annulus $\mathcal{C}_n = \{ \xi \in \mathbb{R}^N \mid |\xi| \leq n \}$. J_n denotes the Friedrichs projector maps L^2 to L_n^2 , defined by

$$\mathcal{F} J_n u(\xi) = 1_{\mathcal{C}_n} \mathcal{F} u(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

We aim to solve the system of ordinary differential equations

$$\begin{cases} \frac{d}{dt} a = F_n(a, \bar{u}, E), \\ \frac{d}{dt} \bar{u} = G_n(a, \bar{u}, E), \\ \frac{d}{dt} E = H_n(a, \bar{u}, E), \\ (a, \bar{u}, E)|_{t=0} = (J_n a_0, 0, J_n E_0), \end{cases} \quad (4.14)$$

in $L_n^2 \times (L_n^2)^N \times (L_n^2)^{N^2}$ with

$$\begin{aligned} F_n(a, \bar{u}, E) &= -J_n(u \cdot \nabla a), \\ G_n(a, \bar{u}, E) &= -J_n(u \cdot \nabla u) + \mu J_n(b \Delta \bar{u}) - J_n(a \nabla \Pi_L) \\ &\quad + \mu J_n(a \Delta u_L) + J_n(b \operatorname{div}(E E^\top)) + J_n(b \operatorname{div} E) \\ &\quad + J_n(b \mathcal{H}_b(-J_n(u \cdot \nabla u) + \mu J_n(a \Delta(\bar{u} + u_L))) \\ &\quad + J_n(b \mathcal{H}_b(a \nabla \Pi_L)) + J_n(b \mathcal{H}_b(b \operatorname{div}(E E^\top) + b \operatorname{div} E)), \\ H_n(a, \bar{u}, E) &= -J_n(u \cdot \nabla E) + J_n(\nabla u \cdot E) + J_n \nabla u. \end{aligned}$$

Here $u = \bar{u} + u_L$, u_L is the solution of (4.1). \mathcal{H}_b denotes the linear operator $F \mapsto \nabla \Pi$, i.e. $\nabla \Pi = \mathcal{H}_b(F)$ is the solution of the elliptic equation

$$\operatorname{div}(b \nabla \Pi) = \operatorname{div} F.$$

The map

$$(a, \bar{u}, E) \mapsto (F_n(a, \bar{u}, E), G_n(a, \bar{u}, E), H_n(a, \bar{u}, E))$$

is locally Lipschitz with respect to the variables (a, \bar{u}, E) . Then we can conclude that the ordinary differential equations has a unique solution (a^n, \bar{u}^n, E^n) in the space $C^1([0, T_n^*]; L_n^2)$. T_n^* is the maximum existence time of (a^n, \bar{u}^n, E^n) . Then using the elliptic equation we can get the existence of $\nabla \Pi^n \in C^1([0, T_n^*]; L_n^2)$.

Now we want to prove that T_n^* may be bounded from below by the supremum T of all the times satisfying (4.13), and that (a^n, \bar{u}^n, E^n) is uniformly bounded in $X_T^{\frac{N}{2}}$. Since J_n is an L^2 orthogonal projector, it has no effect on the priori estimates which were obtained in Section 4.1. Hence, the priori estimates applies to our approximate solution $(a^n, \bar{u}^n, E^n, \nabla \Pi^n)$ which independent of n . And the estimate in (4.5) to $(a^n, \bar{u}^n, E^n, \nabla \Pi^n)$ ensue that it is bounded in $L^\infty(0, T; L_n^2)$. So the standard continuation criterion for ordinary differential equations implies that T_n^* is greater than any time T satisfying (4.13) and for all $n \geq 1$,

$$\begin{cases} \|a^n\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq 2\|a_0\|_{B_{2,1}^{\frac{N}{2}}}, \\ A_T^{\kappa+1} \|a^n - S_{N_0} a^n\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \min\{\frac{1}{4C}b, \frac{1}{4C\mu}\mu\}, \\ \|E^n\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq 6\|E_0\|_{B_{2,1}^{\frac{N}{2}}}, \\ \|\bar{u}^n\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} + \mu\|\bar{u}^n\|_{\tilde{L}_T^1(B_{2,1}^{\frac{N}{2}+1})} + \|\nabla \Pi^n\|_{\tilde{L}_T^1(B_{2,1}^{\frac{N}{2}-1})} \leq \lambda \tilde{U}_0. \end{cases} \quad (4.15)$$

4.3 Compactness arguments

We now have to prove the convergence of (a^n, \bar{u}^n, E^n) . This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 4.1. (a^n, \bar{u}^n, E^n) is uniformly bounded in

$$C^{\frac{1}{2}}([0, T], B_{2,1}^{\frac{N}{2}-1}) \times C^{\frac{1}{2}}([0, T], B_{2,1}^{\frac{N}{2}-2} + B_{2,1}^{\frac{N}{2}-1}) \times C^{\frac{1}{2}}([0, T], B_{2,1}^{\frac{N}{2}-1})$$

for $N \geq 3$.

Proof. We first prove that $\partial_t a^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-1})$, which yields the desired result for a^n .

We observe that a^n satisfies

$$\partial_t a^n = -J_n(u^n \cdot \nabla a^n).$$

According to the uniformly estimates in Section 4.2, \bar{u}^n is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}})$ and a^n is uniformly bounded in $L_T^\infty(B_{2,1}^{\frac{N}{2}})$. The definition of u_L^n obviously provides us with uniform bounds for it in $L_T^2(B_{2,1}^{\frac{N}{2}})$. So we can conclude that

$$\|\partial_t a^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})} \leq C \left(\|\bar{u}^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})} + \|u_L^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})} \right) \|a^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})},$$

which implies that $\partial_t a^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-1})$.

Similarly, we show that $\partial_t E^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-1})$. Let us recall that

$$\partial_t E^n = J_n \left(-(\bar{u}^n + u_L^n) \cdot \nabla E^n + (\nabla \bar{u}^n + \nabla u_L^n) \cdot E^n + (\nabla \bar{u}^n + \nabla u_L^n) \right).$$

By the continuity of Paraproduct in Besov spaces which is stated in Proposition 2.1, we have

$$\|\partial_t E^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})} \leq C \left(\|\bar{u}^n + u_L^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})} \right) \left(\|E^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + 1 \right),$$

which implies that $\partial_t E^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-1})$.

Now we turn to prove $\partial_t \bar{u}^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-2}) + L_T^2(B_{2,1}^{\frac{N}{2}-1})$. Note that $\partial_t \bar{u}^n$ satisfies

$$\begin{aligned} \partial_t \bar{u}^n = J_n \Big(& G_n - \nabla \Pi^n (1 + a^n) + \mu \Delta \bar{u}^n (1 + a^n) - \nabla \Pi_L^n \\ & - (\bar{u}^n + u_L^n) \cdot \nabla \bar{u}^n + \mu a^n \Delta u_L^n - (\bar{u}^n + u_L^n) \cdot \nabla u_L^n \Big), \end{aligned}$$

where $G_{n,i} = (a^n + 1)(\partial_j E_{ik}^n E_{jk}^n + \partial_j E_{ij}^n)$. As the above estimates, we know that

$$\mu \Delta \bar{u}^n (1 + a^n) - (\bar{u}^n + u_L^n) \cdot \nabla \bar{u}^n + \mu a^n \Delta u_L^n - (\bar{u}^n + u_L^n) \cdot \nabla u_L^n + \nabla \Pi_L^n$$

is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-2})$. By the expression of $G_{n,i}$, we have

$$\begin{aligned} \|G_n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})} &\leq CT^{\frac{1}{2}} \left(\|a^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \|E^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})}^2 + \|E^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \right. \\ &\quad \left. + \|a^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \|E^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + \|E^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})}^2 \right), \end{aligned}$$

which implies that G_n is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-1})$. Now we devote to estimate $\nabla \Pi^n$. We split $\nabla \Pi^n$ into $\nabla \Pi_1^n$ and $\nabla \Pi_2^n$, and their satisfy

$$\operatorname{div}(b^n \nabla \Pi_1^n) = \operatorname{div} G_n, \quad \operatorname{div}(b^n \nabla \Pi_2^n) = \operatorname{div} F_n,$$

where $F_n = \mu \Delta \bar{u}^n a^n - (\bar{u}^n + u_L^n) \cdot \nabla \bar{u}^n + \mu a^n \Delta u_L^n - (\bar{u}^n + u_L^n) \cdot \nabla u_L^n$. By the estimate for the elliptic equation in Proposition 3.2, we get

$$\|\nabla \Pi_1^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})} \leq C \|G_n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})}, \quad \|\nabla \Pi_2^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-2})} \leq C \|F_n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-2})}.$$

From the above discuss, we know that $\|G_n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-1})}$ and $\|F_n\|_{L_T^2(B_{2,1}^{\frac{N}{2}-2})}$ are bounded, and $\nabla \Pi^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-2}) + L_T^2(B_{2,1}^{\frac{N}{2}-1})$. So $\partial_t \bar{u}^n$ is uniformly bounded in $L_T^2(B_{2,1}^{\frac{N}{2}-2}) + L_T^2(B_{2,1}^{\frac{N}{2}-1})$. Thus we have proved the lemma. \square

We can now turn to prove the existence of a solution. The procedure is similar as been used in [7]. We can see in [7] that the approximation solutions are convergence in the term of subsequence by Ascoli's theorem. So we omit the details. The same argument to $N = 2$, we can also prove that $\partial_t \bar{u}^n$ is uniformly bounded in $L^{\frac{4}{3}}([0, T], B_{2,1}^{-\frac{1}{2}})$.

4.4 Uniqueness

Assume that we have two solutions of (1.5), $(a_1, u_1, E_1, \nabla \Pi_1)$ and $(a_2, u_2, E_2, \nabla \Pi_2)$ with the same initial data satisfying the regularity assumptions of Theorem 1.1. We first consider the case $N \geq 3$.

Set $a_1 - a_2 = \delta a$, $u_1 - u_2 = \delta u$, $\nabla \Pi_1 - \nabla \Pi_2 = \nabla \delta \Pi$, $E_1 - E_2 = \delta E$. Then $(\delta a, \delta u, \nabla \delta P, \delta E)$ satisfies the following system

$$\begin{cases} \partial_t \delta a + u_2 \cdot \nabla \delta a = -\delta u \cdot \nabla a_1, \\ \partial_t \delta u + u_2 \cdot \nabla \delta u - \mu(1 + a_1)(\Delta \delta u - \nabla \delta \Pi) = \delta G + \delta H, \\ \partial_t \delta E + u_2 \cdot \nabla \delta E = \delta L, \\ \operatorname{div} \delta u = 0, \\ (\delta a, \delta u, \delta E)|_{t=0} = (0, 0, 0), \end{cases} \quad (4.16)$$

where

$$\begin{aligned} \delta H &= -\delta u \cdot \nabla u_1 + \mu \delta a \Delta u_2 - \delta a \nabla \Pi_2, \\ \delta G_i &= (a_1 + 1) \partial_j \delta E_{ik} E_{1,jk} + (a_1 + 1) \partial_j E_{2,ik} \delta E_{jk} \\ &\quad + \delta a \partial_j E_{2,ik} E_{2,jk} + \delta a \partial_j E_{2,ij} + (a_1 + 1) \partial_j \delta E_{ij}, \\ \delta L &= -\delta u \cdot \nabla E_1 + \nabla u_2 \cdot \delta E + \nabla \delta u \cdot E_1 + \nabla \delta u. \end{aligned}$$

From Proposition 3.1, we have

$$\begin{aligned} \|\delta a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} &\leq C_T \|a_1 - S_{N_0} a_1\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})} \\ &\quad + C_T \|S_{N_0} a_1\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}+\eta})} T^{\frac{\eta}{2}} \|\delta u\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}-2})}^{\frac{\eta}{2}} \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})}^{1-\frac{\eta}{2}}, \end{aligned} \quad (4.17)$$

where $V(T) = \|\nabla u_2\|_{L_T^1(B_{2,1}^{\frac{N}{2}})}$ and $\eta \in (0, 1)$. Similarly, we have

$$\begin{aligned} \|\delta E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} &\leq e^{CV(T)} \left(\int_0^T \|u_2\|_{B_{2,1}^{\frac{N}{2}+1}} \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} dt \right. \\ &\quad \left. + \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})} \|E_1\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} + \|\nabla \delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}-1})} \right) \\ &\leq C_T \left(\|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})} + \int_0^T \|u_2\|_{B_{2,1}^{\frac{N}{2}+1}} \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} dt \right). \end{aligned} \quad (4.18)$$

Next, denoting $\delta U = \|\delta u\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-2})} + \underline{\mu} \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})} + \|\nabla \delta \Pi\|_{L_T^1(B_{2,1}^{\frac{N}{2}-2})}$, the estimate for linear momentum equations, Proposition 3.2 guides us to get

$$\begin{aligned} \delta U &\leq e^{CV(T)} A_{1,T}^\kappa \left(\int_0^T \|\delta u\|_{B_{2,1}^{\frac{N}{2}-2}} \|\nabla u_1\|_{B_{2,1}^{\frac{N}{2}}} + \mu \|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} \|\Delta u_2\|_{B_{2,1}^{\frac{N}{2}-1}} dt \right. \\ &\quad \left. + \int_0^T \|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} \|\nabla \Pi_2\|_{B_{2,1}^{\frac{N}{2}-1}} dt + \|\delta G\|_{L_T^1(B_{2,1}^{\frac{N}{2}-2})} \right. \\ &\quad \left. + \mu A_{1,T} \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}-\alpha})} \right), \end{aligned} \quad (4.19)$$

with $A_{1,T} = 1 + \underline{b} 2^{N_0 \alpha} \|a_1\|_{B_{2,1}^{\frac{N}{2}}}$ and $\alpha \in (0, 1)$. By interpolation, the last term can be bounded by

$$\|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}-\alpha})} \leq C \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}-2})}^{\frac{\alpha}{2}} \|\delta u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})}^{1-\frac{\alpha}{2}}.$$

Young's inequality and the uniform bounded of the solution imply that

$$\begin{aligned} \delta U &\leq C_T \int_0^T (1 + \|u_2\|_{B_{2,1}^{\frac{N}{2}+1}} + \|\nabla \Pi_2\|_{B_{2,1}^{\frac{N}{2}+1}}) \\ &\quad \left(\|\delta u\|_{B_{2,1}^{\frac{N}{2}-2}} + \|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} dt + \|\delta G\|_{L_T^1(B_{2,1}^{\frac{N}{2}-2})} \right) dt. \end{aligned}$$

By the expression of δG , we have the following estimate

$$\begin{aligned}
& \|\delta G\|_{L_T^1(B_{2,1}^{\frac{N}{2}-2})} \\
& \leq \int_0^T \left(\|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} \|E_2\|_{B_{2,1}^{\frac{N}{2}}}^2 + (\|E_1\|_{B_{2,1}^{\frac{N}{2}}} + \|E_2\|_{B_{2,1}^{\frac{N}{2}}}) \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} \right. \\
& \quad + \|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} \|E_2\|_{B_{2,1}^{\frac{N}{2}}} + \|a_1\|_{B_{2,1}^{\frac{N}{2}}} \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} \\
& \quad \left. + \|a_1\|_{B_{2,1}^{\frac{N}{2}}} (\|E_1\|_{B_{2,1}^{\frac{N}{2}}} + \|E_2\|_{B_{2,1}^{\frac{N}{2}}}) \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} \right) dt \\
& \leq C_T \int_0^T \left(\|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} + \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} \right) dt.
\end{aligned}$$

Combination the above estimates, we know that

$$\begin{aligned}
& \|\delta a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} + \|\delta E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} + \delta U \\
& \leq C_T \int_0^T (1 + \|u_2\|_{B_{2,1}^{\frac{N}{2}+1}} + \|\nabla \Pi_2\|_{B_{2,1}^{\frac{N}{2}+1}}) \\
& \quad \times \left(\|\delta u\|_{B_{2,1}^{\frac{N}{2}-2}} + \|\delta a\|_{B_{2,1}^{\frac{N}{2}-1}} + \|\delta E\|_{B_{2,1}^{\frac{N}{2}-1}} \right) dt.
\end{aligned}$$

which yields

$$\|\delta a\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} = \|\delta E\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}-1})} = \delta U = 0,$$

for small enough T . A standard continuity argument allows us to know the uniqueness on $[0, T^*)$, T^* is the lifespan of the local solution. This finish the proof of the uniqueness of Theorem 1.1 when $N \geq 3$.

In the case of $N = 2$, the above proof fails because $\frac{N}{2} - 1 = 0$. Hence we may be tempted to estimate $(\delta a, \delta u, \delta E, \nabla \delta \Pi)$ in

$$L_T^\infty(B_{2,\infty}^0) \times L_T^\infty(B_{2,\infty}^{-1}) \cap \tilde{L}_T^1(B_{2,\infty}^1) \times L_T^\infty(B_{2,\infty}^0) \times \tilde{L}_T^1(B_{2,\infty}^{-1}).$$

Now we give the details of the proof. From Proposition 3.1, we have

$$\begin{aligned}
\|\delta a\|_{\tilde{L}_T^\infty(B_{2,\infty}^0)} & \leq e^{CV(T)} \left(\|\delta u\|_{\tilde{L}_T^1(B_{2,1}^1)} \|a_1\|_{\tilde{L}_T^\infty(B_{2,1}^1)} \right) \\
& \leq C_T \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \log \left(e + \frac{\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^2)}}{\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)}} \right).
\end{aligned} \tag{4.20}$$

When T is finite, $\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^2)} \leq W(T)$, where $W(T)$ is finite. Hence

$$\|\delta a\|_{\tilde{L}_T^\infty(B_{2,\infty}^0)} \leq C_T \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \log \left(e + \frac{W(T)}{\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)}} \right). \tag{4.21}$$

Also we can get

$$\begin{aligned}
& \|\delta E\|_{\tilde{L}_T^\infty(B_{2,\infty}^0)} \\
& \leq e^{CV(T)} \left(\|\delta u\|_{\tilde{L}_T^1(B_{2,1}^1)} \|E_1\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})} + \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \right. \\
& \quad \left. + \int_0^T \|u_2\|_{B_{2,1}^2} \|\delta E\|_{B_{2,\infty}^0} dt \right) \\
& \leq C_T \left(\|\delta u\|_{\tilde{L}_T^1(B_{2,1}^1)} + \int_0^T \|u_2\|_{B_{2,1}^2} \|\delta E\|_{B_{2,\infty}^0} dt \right)
\end{aligned} \tag{4.22}$$

$$\leq C_T \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \log \left(e + \frac{W(T)}{\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)}} \right) + \int_0^T \|u_2\|_{B_{2,1}^2} \|\delta E\|_{B_{2,\infty}^0} dt.$$

Then Gronwall inequality implies

$$\|\delta E\|_{\tilde{L}_T^\infty(B_{2,\infty}^0)} \leq C_T \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \log \left(e + \frac{W(T)}{\|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)}} \right). \quad (4.23)$$

Next, denoting $\delta U = \|\delta u\|_{L_T^\infty(B_{2,\infty}^{-1})} + \underline{\mu} \|\delta u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_T^1(B_{2,\infty}^{-1})}$, the estimate for linear momentum equations, Proposition 3.2 guides us to get

$$\begin{aligned} \delta U &\leq e^{C^V(T)} A_{1,T}^\kappa \left(\int_0^T \|\delta u\|_{B_{2,\infty}^{-1}} \|\nabla u_1\|_{B^2} + \mu \|\delta a\|_{B_{2,\infty}^0} \|u_2\|_{B^2} dt \right. \\ &\quad \left. + \int_0^T \|\delta a\|_{B_{2,\infty}^0} \|\nabla \Pi_2\|_{B_{2,1}^0} dt + \|\delta G\|_{\tilde{L}_T^1(B_{2,\infty}^{-1})} \right. \\ &\quad \left. + \mu A_{1,T} \|\delta u\|_{L_T^1(B_{2,1}^{1-\alpha})} \right), \end{aligned} \quad (4.24)$$

with $A_{1,T} = 1 + \underline{b} 2^{N_0 \alpha} \|a_1\|_{\tilde{L}_T^\infty(B_{2,1}^1)}$ and $\alpha \in (0, 1)$. By the expression of δG , with the same calculus as $N \geq 3$, we have the following estimate

$$\|\delta G\|_{\tilde{L}_T^1(B_{2,\infty}^{-1})} \leq C_T \int_0^T \left(\|\delta a\|_{B_{2,\infty}^0} + \|\delta E\|_{B_{2,\infty}^0} \right) dt. \quad (4.25)$$

Hence combination the above estimates, together with interpolation and Young's inequality, we know that

$$\delta U(T) \leq C_T \int_0^T \left((1 + \|u_1(t)\|_{B_{2,1}^2}) \delta U(t) \log \left(e + \frac{W(T)}{\delta U(t)} \right) \right) dt \quad (4.26)$$

which yields $\delta U = 0$ on $[0, T]$ by Osgood Lemma (Lemma 3.4, Chapter 3, [1]) because

$$\int_0^1 \frac{1}{r \log \left(e + \frac{C}{r} \right)} dr = +\infty.$$

This finish the proof of the uniqueness of Theorem 1.1 when $N = 2$.

5 The Global Theory For Small Initial Velocity

In the above section, we have proved that there exists a unique local solution (a, u, E) of (1.5) in $C([0, T]; B_{2,1}^{\frac{N}{2}}) \times X_T^{\frac{N}{2}-1}$. We have used the L^2 estimate for $\nabla \Pi$ in (3.13). That is the reason why we work on the nonhomogeneous Besov space. We rewrite (3.13) as follows:

$$\underline{b} \|\nabla \Pi\|_{B_{2,1}^\sigma} \lesssim \|\mathcal{Q}L\|_{B_{2,1}^\sigma} + \|a\|_{B_{2,1}^{\frac{N}{2}}} \|\nabla \Pi\|_{B_{2,1}^\sigma}$$

While fortunately, the assumption on a_0 in Theorem 1.2 can avoid the L^2 estimate of $\nabla \Pi$. More precisely, the second term can be absorbed by the left hand side due to the smallness condition on a . Thus using the same method as in Theorem 1.1, we obtain that there exists a unique local solution (a, u, E) of (1.5) in

$$C([0, T^*]; \dot{B}_{2,1}^{\frac{N}{2}}) \times L^1(0, T^*; \dot{B}_{2,1}^{\frac{N}{2}+1}) \cap C([0, T^*]; \dot{B}_{2,1}^{\frac{N}{2}-1}) \times C([0, T^*]; \dot{B}_{2,1}^{\frac{N}{2}}),$$

where T^* is the maximum existence time of (a, u, E) . From the assumptions in Theorem 1.4, using Proposition 3.1, we can easily obtain

$$a \in C([0, T^*]; \dot{B}_{2,1}^{\frac{N}{2}-1}), \quad E \in C([0, T^*]; \dot{B}_{2,1}^{\frac{N}{2}-1}).$$

We define $d^{ij} = -\Lambda^{-1}\partial_j u^i$, then $u^i = \Lambda^{-1}\partial_j d^{ij}$. Applying $-\Lambda^{-1}\partial_j$ to the second equation of system (1.5), we get

$$\begin{aligned} \partial_t d^{ij} + u \cdot \nabla d^{ij} - \mu \Delta d^{ij} &= -u \cdot \nabla (\Lambda^{-1}\partial_j u^i) \\ &\quad + \Lambda^{-1}\partial_j \left(u \cdot \nabla u^i + (a+1)\partial_i \Pi - \mu a \Delta u^i - G_i \right), \end{aligned} \quad (5.1)$$

where $G_i = (a+1)(\partial_j E_{ik} E_{jk} + \partial_j E_{ij})$ is defined in Section 1. Note that the compatibility condition (1.3), we have

$$\begin{aligned} \Lambda^{-1}\partial_j \partial_k E_{ik} &= \Lambda^{-1}\partial_k \partial_j E_{ik} \\ &= \Lambda^{-1}\partial_k (\partial_k E_{ij} + E_{lk} \partial_l E_{ij} - E_{lj} \partial_l E_{ik}) \\ &= -\Lambda E_{ij} + \Lambda^{-1}\partial_k (E_{lk} \partial_l E_{ij} - E_{lj} \partial_l E_{ik}). \end{aligned} \quad (5.2)$$

Combination (5.1), (5.2) with (1.5) yields

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t d^{ij} + u \cdot \nabla d^{ij} - \mu \Delta d^{ij} - \Lambda E_{ij} = H, \\ \partial_t E + u \cdot \nabla E + \Lambda d^{ij} = R, \\ d^{ij} = -\Lambda^{-1}\partial_j u^i, \\ \operatorname{div} u = 0, \\ (a, u, E)|_{t=0} = (a_0, u_0, E_0), \end{cases} \quad (5.3)$$

where

$$\begin{aligned} H &= -u \cdot \nabla (\Lambda^{-1}\partial_j u^i) + \Lambda^{-1}\partial_j \left(u \cdot \nabla u^i + (a+1)\partial_i \Pi - \mu a \Delta u^i \right. \\ &\quad \left. - (a+1)\partial_l E_{ik} E_{lk} - a \partial_l E_{il} \right) - \Lambda^{-1}\partial_k (E_{lk} \partial_l E_{ij} - E_{lj} \partial_l E_{ik}), \\ R &= \partial_k u^i E_{kj}. \end{aligned}$$

Denote $\alpha = \|a_0\|_{\tilde{B}_{\mu}^{\frac{N}{2}, \infty}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} + \|E_0\|_{\tilde{B}_{\mu}^{\frac{N}{2}, \infty}}$. We are going to prove the existence of a positive M such that, if α is small enough, the following bound holds

$$\|a\|_{\tilde{L}^\infty(\tilde{B}_{\mu}^{\frac{N}{2}, \infty})} + \|(u, E)\|_{Y_{T^*}^{\frac{N}{2}}} \leq M\alpha. \quad (5.4)$$

This estimate is the direct product of the following proposition.

Proposition 5.1. *If*

$$\|a\|_{\tilde{L}_T^\infty(\tilde{B}_{\mu}^{\frac{N}{2}, \infty})} + \|(u, E)\|_{Y_T^{\frac{N}{2}}} \leq 2M\alpha, \quad T \in (0, T^*),$$

then, we have

$$\|a\|_{\tilde{L}_T^\infty(\tilde{B}_{\mu}^{\frac{N}{2}, \infty})} + \|(u, E)\|_{Y_T^{\frac{N}{2}}} \leq M\alpha,$$

when α is small enough.

Proof. First, from Proposition 3.1 in [9], we obtain

$$\|a\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{N}{2}})} \leq C_1 e^{\tilde{V}(T)} \|a_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}},$$

$$\|a\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{N}{2}-1})} \leq C_1 e^{\tilde{V}(T)} \|a_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}},$$

where $\tilde{V}(T) = \int_0^T \|\nabla u\|_{\dot{B}_{2,1}^{\frac{N}{2}}} dt$. If we assume α small enough such that

$$e^{2M\alpha} \leq 2,$$

then we have

$$\|a\|_{\tilde{L}_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} \leq M\alpha$$

for $M = 4C_1$. From Proposition 3.4, we have

$$\begin{aligned} \|(d, E)\|_{Y_T^{\frac{N}{2}}} &\leq C e^{V(T)} \left(\|E_0\|_{\tilde{B}_\mu^{\frac{N}{2},\infty}} + \|d_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \right. \\ &\quad \left. + \|R\|_{L_T^1(\tilde{B}_\mu^{\frac{N}{2},\infty})} + \|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} \right). \end{aligned} \quad (5.5)$$

We want to bound $\|R\|_{L_T^1(\tilde{B}_\mu^{\frac{N}{2},\infty})}$ and $\|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})}$. With the help of Proposition 2.3, we have

$$\|R\|_{L_T^1(\tilde{B}_\mu^{\frac{N}{2},\infty})} \leq C \|E\|_{L_T^1(\tilde{B}_\mu^{\frac{N}{2},\infty})} \|\nabla u\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}})} \leq CM^2 \alpha^2.$$

Now, we devote to estimate $\|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})}$. From the expression of H , the trouble is the estimate for $\nabla \Pi$. Applying div to the momentum equation of (1.5) yields

$$\partial_i((a+1)\partial_i \Pi) = \partial_i(-u \cdot \nabla u^i + \mu a \Delta u^i + L_i),$$

with

$$L_i = (a+1)\partial_j E_{ik} E_{jk} + a\partial_j E_{ij}.$$

Here we have used $\operatorname{div}(E^\top) = 0$. Then by the estimate of elliptic equation, the following bound holds

$$\|\nabla \Pi\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} \leq C \|u\|_{L_T^2(\dot{B}_{2,1}^{\frac{N}{2}})}^2 + C \|a\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{N}{2}})} \|u\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}+1})} + \|L_i\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})}.$$

Note the expression of L_i , we only need to estimate $\|a\partial_j E_{ij}\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})}$. In fact, by Proposition 2.4 with $s = \frac{N}{2}, t = \frac{N}{2} - 1$, we have the following

$$\begin{aligned} \|a\partial_j E_{ij}\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} &\leq C \|a\|_{L_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} \|E\|_{L_T^1(\tilde{B}_\mu^{\frac{N}{2},1})} \\ &\leq CM^2 \alpha^2. \end{aligned} \quad (5.6)$$

Others in L_i are estimated similarly. Hence we get

$$\|\nabla \Pi\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} \leq CM^2 \alpha^2 (1 + M\alpha).$$

For the term of H , using the estimate of $\nabla \Pi$ and Proposition 2.2, we have

$$\begin{aligned} \|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} &\leq C \|u\|_{L_T^2(\dot{B}_{2,1}^{\frac{N}{2}})}^2 + C\mu \|a\|_{L_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} \|u\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}+1})} \\ &\quad + C \|a\|_{L_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} \|\nabla \Pi\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} + C \|\nabla \Pi\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} \\ &\quad + C \|(a+1)E \cdot \nabla E\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} + \|a\nabla E\|_{L_T^1(\dot{B}_{2,1}^{\frac{N}{2}-1})} \\ &\leq CM^2 \alpha^2 (1 + M\alpha). \end{aligned} \quad (5.7)$$

Plugging the estimates on H and R into (5.5), noting that $d^{ij} = -\Lambda^{-1}\partial_j u^i$, we have

$$\begin{aligned} \|a\|_{\tilde{L}_T^\infty(\tilde{B}_\mu^{\frac{N}{2},\infty})} + \|(u, E)\|_{Y_T^{\frac{N}{2}}} &\leq C_2 e^{2M\alpha} \left(\alpha + C_2 M^2 \alpha^2 + C_2 M^3 \alpha^3 \right) \\ &\leq M\alpha, \end{aligned} \quad (5.8)$$

when $M = 4C_2$ and α satisfies

$$e^{2M\alpha} \leq 2, \quad 2C_2^2 M\alpha \leq \frac{1}{4}, \quad 2C_2^2 M^2 \alpha^2 \leq \frac{1}{4}.$$

Then, we finish the proof of Proposition 5.1 for $M = \max\{4C_1, 4C_2\}$. \square

Now we can give the proof of the global existence. From the standard continuation method and Proposition 5.1, we easily obtain that (5.4) holds. Combining the local existence, if T^* is finite, then the lifespan of the solution is greater than T^* . Hence $T^* = \infty$ and we finish the proof of Theorem 1.2.

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